VARIATION IN THE DISSIPATION OF ENERGY IN A TURBULENT FLOW AND THE SPECTRAL DISTRIBUTION OF ENERGY

(IZMENCHIVOST' DISSIPATSII ENERGII V TURBULENTNOM POTOKE I RASPREDELENIE ENERGII PO SPEKTRU)

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1. The dissipation of kinetic energy represents one of the fundamental characteristics of a turbulent flow

$$\varepsilon = 2\nu D_{ik}^2, \qquad D_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \tag{1}$$

where ν is the kinematic viscosity, v_i the velocity field and D_{ik} the deformation tensor of a particle of liquid. The quantity ε, as well as the velocity, depends on the coordinates and time in a random fashion.

In the work of Kolmogorov [1], two similarity hypotheses were introduced, concerning the structure of turbulent flow at large Reynolds numbers. According to the first hypothesis, the structure of a turbulent flow, on a scale sufficiently small in comparison to the characteristic external scale of turbulence L, is determined by two parameters; the average dissipation of energy $\langle \varepsilon \rangle$, and ν. According to the second hypothesis, in the so-called inertial distance interval

$$L \gg r \gg l_0 \equiv \nu^{3/4} \langle \varepsilon \rangle^{-1/4} \tag{2}$$

(l_0 is the internal scale of the turbulence [1]), essentially only the parameter $\langle \varepsilon \rangle$ remains. Based on the hypothesis of similarity, a series of results were obtained, the most important one being the "2/3 law" of Kolmogorov-Obukhov [1,2] and its analogy in spectral terminology, the "5/3 law"

$$E(p) = C \langle \varepsilon \rangle^{2/3} p^{-5/3} \; (l_0^{-1} \gg p \gg L^{-1}) \tag{3}$$

where $E(p)$ is the spectral density of the kinetic energy, p is the wave number and C is a constant.

Soon after the origin of the similarity concept of turbulence, Landau [3] pointed out the necessity of taking into account the variations of ε, related to macroscopic movement. The marked refinement of the similarity hypothesis taking into account the statistical nature (variability) of the dissipation of energy in a turbulent flow is discussed in [4] and [5]. In a somewhat different manner, the same idea of taking into account the variability of the dissipation of energy is used in the present article, which leads to the following, at first glance paradoxical, results: (a) similarity generally "worsens" in proportion to the penetration into the smaller scales; (b) the velocity field, sufficiently smooth (in the sense of the existence of higher derivatives), in individual applications exhibits peaks after statistical averaging (the mean square values of sufficiently high derivatives go to infinity).

2. In [4,5] is introduced the concept of a "pure" statistical ensemble with a fixed value for $\varepsilon(r)$, which is the energy dissipation, averaged over a sphere of radius r. The similarity hypothesis for the "pure" ensemble is formulated and the expression for the structural functions for the velocity field is written in function of the distance r. For the "mixed" (complete) ensemble, the corresponding equations are obtained by averaging over different values of $\varepsilon(r)$, for which [4,5] one takes the logarithmic-normal law of distribution with dispersion depending on the distance r.

In the present article, it will be more convenient to use a somewhat different definition of a "pure" ensemble. For describing turbulence in fixed (Eulerian) coordinates, the zones, variably saturated by dissipation, rush past the observer who is measuring the velocity at fixed points. It is natural to assume that the structural functions of the velocity field, corresponding to the distance r, are determined by energy dissipation averaged over a region with dimensions much larger than r, so that it is possible to "accumulate statistics".* By "pure" will be understood the ensemble with the fixed quantity ε (the energy dissipation), averaged over the region with dimensions sufficiently large compared with the distance r in which we are interested. The "mixed" ensemble is the superposition of "pure" ensembles with a certain (independent of r) statistical distribution, satisfying the following condition of orthogonality:

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* If the structural functions of multiple points are considered, then r is understood to be the distance between points with maximum separation.

$$\int_0^\infty f(\varepsilon)\, d\varepsilon = 1 \tag{4}$$

It is possible that analogous concepts were implied in [6], where the authors reached the conclusion that the variability of energy dissipation mildly affects the form of the distribution of kinetic energy over the spectrum, including the region of large wave numbers. However, such a conclusion is related only with a special choice of form of spectral energy density for a "pure" ensemble: two exponential laws, matched in the region of the interior scale of turbulence. It is shown below that in other initial spectra, for a "pure" ensemble, the variability of energy dissipation substantially affects the form of spectral energy in a "mixed" ensemble, especially in the region of large wave numbers.

3. From considerations of similarity, the following expression for the spectrum of energy in the "pure" ensemble can be written:

$$E(p) = \varepsilon^{2/3} p^{-5/3} \varphi(p^2 \nu \tau) \qquad (\tau = \nu^{1/2} \varepsilon^{-1/2},\ pL \gg 1) \tag{5}$$

The function $\varphi(x)$, which may be considered to be universal (independent of the macroscopic properties of the flow), satisfies the following conditions:

$$\int_0^\infty x^{-1/3} \varphi(x)\, dx = 1, \qquad \lim_{x\to 0} \varphi(x) = C \tag{6}$$

The first condition is derived from the normalization of the spectrum

$$2\nu \int_0^\infty p^2 E(p)\, dp = \varepsilon \tag{7}$$

The second condition points out that, in the inertial interval of wave numbers, expression (5) must transform into the "5/3 law".

For the "mixed" ensemble, averaging (5) with respect to ε weighted by $f(\varepsilon)$, we obtain

$$\langle E(p)\rangle = \langle\varepsilon\rangle^{2/3} p^{-5/3} \varphi_*(p^2 \nu \tau_0) \qquad (\tau_0 = \nu^{1/2} \langle\varepsilon\rangle^{-1/2},\ pL \gg 1)$$
$$\varphi_*(p^2 \nu \tau_0) = \langle \varepsilon^{2/3} \varphi(p^2 \nu \tau)\rangle \langle\varepsilon\rangle^{-2/3}, \qquad C_* = \lim_{x\to 0} \varphi_*(x) = C \langle\varepsilon^{2/3}\rangle \langle\varepsilon\rangle^{-2/3} \tag{8}$$

(the function $\varphi_*(x)$ satisfies the first condition in (6)). The distribution $f(\varepsilon)$ depends upon macroscopic properties of the flow, and, in this respect, the function $\varphi_*(x)$ and the spectrum $\langle E(p)\rangle$ do not appear to be universal. It can be observed from (8) that the "5/3 law" always occurs in the inertial interval of wave numbers; however, the constant C_* may depend upon macroscopic properties of the flow.* Further, if the spectrum in the pure ensemble has an exponential asymptote in the region of large wave numbers, then this same asymptote (up to a nonuniversal constant) will also have spectra in the "mixed" ensemble. This case, too, was considered in [6] above, where the exponent in the asymptote is taken to be equal to −7 (Heisenberg's modulus [8]). It seems more natural to assume that in the "pure" ensemble, the velocity field has in the mean-square enough high derivatives. Then the spectrum in the "pure" ensemble must decrease in the region of large wave numbers faster than an arbitrary exponent; and the spectrum in the "mixed" ensemble will not have universal asymptotic behavior in the region of large wave numbers (even up to a constant). In this respect there occurs derivation (a) formulated at the end of Section 1. Derivation (b) follows from the fact that the function $\varphi_*(x)$ may have an exponential asymptote even if $\varphi(x)$ decreases faster than any power (or $\varphi_*(x)$ may decrease with a smaller exponent). Both of these derivations are illustrated below in a concrete example.

Note that in some ranges of the variable ε (for a constant average value $\langle\varepsilon\rangle$) the distribution $f(\varepsilon)$ may possess universality. Considering this, derivation (a) should be formulated more cautiously; the dependence of the characteristics of turbulence (especially the energy spectrum) upon the probability distribution for ε is more marked as one reaches ever smaller dimensional scales. The experimental investigation of the distribution function $f(\varepsilon)$ would be interesting.

4. In [9] is derived the asymptotic behavior of an energy spectrum in the region of large wave numbers, which combines in a natural manner, with the "5/3 law"; on the basis of such a combination, the formula for an energy spectrum throughout the interval $pL \gg 1$ was presented in [9]; it may be written in the form

$$E(p) = C\nu^{2/3}\tau^{-4/3}p^{-5/3}\exp\{-ap^2\nu\tau\} \qquad \left(C = \frac{a^{2/3}}{\Gamma(2/3)},\ a \approx \frac{2\sqrt{7}}{3}\right) \tag{9}$$

where $\Gamma(x)$ is the gamma function. We will accept this formula as the initial spectral density in the "pure" ensemble. Instead of the distribution $f(\varepsilon)$, it is more convenient to introduce the distribution of

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* Note that, for the Lagrangean description of turbulence, the second moments of velocity and the distances between fluid particles in the corresponding inertial interval of time depend linearly upon dissipation; therefore the corresponding constants turn out to be universal [7].

the quantity $y = \tau \langle \tau \rangle^{-1}$, which, for the sake of simplicity in calculation, we represent by

$$P(y) = \frac{\alpha^{\alpha+1}}{\Gamma(\alpha+1)} y^{\alpha} e^{-\alpha y} \tag{10}$$

The only parameter characterizing the distribution $P(y)$ and, consequently, the macroscopic properties of turbulent flow, is the quantity α, relating the average value and dispersion of dissipation by the formulas

$$\langle \varepsilon \rangle = \frac{\nu}{\langle \tau \rangle^2} \frac{\alpha}{\alpha - 1}, \qquad \frac{\langle [\varepsilon - \langle \varepsilon \rangle]^2 \rangle}{\langle \varepsilon \rangle^2} = \frac{2(2\alpha - 3)}{(\alpha - 2)(\alpha - 3)} \tag{11}$$

In order for the average value of dissipation to exist, it is necessary that $\alpha > 1$. For the existence of dissipation dispersion (which, in general, is not necessarily finite in the "mixed" ensemble), it is necessary that $\alpha > 3$.

Averaging (9) we get

$$\langle E(p) \rangle = \frac{C_* \langle \varepsilon \rangle^{2/3} p^{-5/3}}{[1 + \beta (p l_0)^2]^{\alpha - 1/3}} \tag{12}$$

$$C_* = \frac{C \Gamma(\alpha - 1/3)}{m^{1/3} \Gamma(\alpha - 1)}, \qquad \beta = a m^{-1/2}, \qquad m = \alpha(\alpha - 1) \tag{13}$$

Equation (12) clearly illustrates how the formulas are derived. Notice that for $\alpha = 3$ a power asymptote of the Heisenberg model (exponent $= -7$) is derived.

BIBLIOGRAPHY

1. Kolmogorov, A.N., Lokal'naia struktura turbulentnosti v neszhimaemoi zhidkosti pri ochen' bol'shikh chislakh Rinol'dsa (Local structure of turbulence in an incompressible fluid at very high Reynolds numbers). *Dokl. Akad. Nauk SSSR*, Vol. 30, No. 4, 1941.

2. Obukhov, A.M., O raspredelenii energii v spektre turbulentnogo potoka (The dispersion of energy in the spectra of a turbulent flow). *Izv. Akad. Nauk SSSR, ser. geogr. i geofiz.*, No. 5, 1941.

3. Landau, L.D. and Livshitz, E.M., *Mekhanika sploshnykh sred (The Mechanics of Continuous Media)*. Gostekhizdat, 1944.

4. Oboukhov, A.M., Some specific features of atmospheric turbulence. *J. Fluid Mech.*, Vol. 13, Part 1, 1962.

5. Kolmogorov, A.N., A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.*, Vol. 13, Part 1, 1962.

6. Grant, H.L., Stewart, R.W. and Moilliet, A., Turbulence spectra from a tidal channel. *J. Fluid Mech.* Vol. 12, Part 2, 1962.

7. Novikov, E.A., Metod sluchainykh sil v teorii turbulentosti (Methods of random forces in the theory of turbulence). *ZhETF* Vol. 44, No. 6, 1963.

8. Heisenberg, W., Zur statistischen Theorie der Turbulenz. *Z. Phys.*, Vol. 124, 1948.

9. Novikov, E.A., O spektre energii turbulentnogo potoka neszhimaemoi zhidkosti (The spectral energy of turbulent flow of an incompressible fluid). *Dokl. Akad. Nauk SSSR*, Vol. 139, No. 2, 1961.

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